

SUPERPOSITION AND TIME DELAY ARE INCOMPATIBLE BETWEEN THEM

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ABSTRACT. We analyse the properties of wave functions $\psi(x, t; y, s)$ that depend at once on two time-position vectors (x, t) and (y, s) with different time entries $t \neq s$. In particular, we prove that the existence of such wave functions is indeed incompatible with the superposition property and the technique of separation of variables.

1. TIME DELAYED WAVE FUNCTIONS

Quantum mechanics is one of the most successful theories of the twenty century; its predictions have been verified with such a remarkable precision, that one is automatically compelled to consider whether the Schrödinger equation can be modified to include different phenomena, like time delays. Many works can be found in the literature on this subject; see for example Muga et al's books [6, 7]. Moreover, in chapter xviii of Böhm's book [1] it is analysed the relation between time delays and quantum scattering; while in chapters 18 and 19 of Razavy's book [9] it is analysed the connection between time delays and quantum tunneling. In any case, we can safely assert that time delays and quantum mechanics seem to have a strange and intriguing relationship, that maybe goes back to 1978, when Wheeler proposed his delayed choice *gedanken* experiment [11, 5]. This relation became even stronger in 2000 and later, when the delayed choice quantum eraser was experimentally verified [4, 2, 10, 8, 3].

Nevertheless, the connections between time delays and quantum mechanics mentioned above seem to be subjective, so that we wonder whether they

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can be formalised; i.e., we wonder whether there could exist a wave function $\psi^*(x, t; y, s)$ that depends at once on two time-position vectors (x, t) and (y, s) with different time entries $t \neq s$. We are in particular interested in the simple case, when the difference $t - s$ is equal to a fixed constant $\tau > 0$, so that the square of the absolute value $|\psi^*(x, t; y, s)|^2$ is the probability density function that the system **is** in the position x at time t and it **was** in the position y at time $s = t - \tau$.

Now then, if such a wave function $\psi^*(x, t; y, s)$ exists, the state of the system would be described by two objects: a classical wave function $\psi(x, t)$ and the new one ψ^* . In particular, the term $i\hbar \frac{\partial}{\partial t} \psi(x, t)$ in the Schrödinger equation should be equal to a Hamiltonian that linearly depends on one or all of the following functions: $\psi(x, t)$, $\psi(y, s)$, and $\psi^*(x, t; y, s)$.

The main objective of this work is to prove that the existence of such a wave function $\psi^*(x, t; y, s)$ is incompatible with the superposition property and the technique of separation of variables; and this incompatibility does not come from the structure of the corresponding Schrödinger equation, but from the fact that $|\psi^*(x, t; y, s)|^2$ is the probability density function that the system is in the state x at time t and it was (or will be) in the state y at time s .

Since $|\psi(x, t)|^2$ is the probability density function that the system is in the state x at time t , its integral with respect to x must be equal to one. In the same way, the integral of $|\psi^*(x, t; y, s)|^2$ with respect to the variable y must be equal to the probability that the system is at state x at time t ; i.e., we have for all times t and s that

$$(1) \quad \int |\psi(x, t)|^2 dx \equiv 1 \quad \text{and}$$

$$(2) \quad \int |\psi^*(x, t; y, s)|^2 dy = |\psi(x, t)|^2.$$

We prove in the following two chapters that the previous pair of identities is indeed incompatible with the superposition property and the technique of separation of variables.

2. SEPARATION OF VARIABLES AND SUPERPOSITION

The fact that the system could be described by a pair of wave functions ψ^* and ψ drives us to think that there should be a relation between them;

i.e. there should exist an operator that transforms the function $\psi^*(x, t; y, s)$ into $\psi(x, t)$ by *forgetting* the dependence on the time-position vector (y, s) . One of the simplest forms to visualise this relation is to use the technique of separation of variables, which asserts that one can find wave functions (solutions to the corresponding Schrödinger equation) of the form

$$(3) \quad \psi_k^*(x, t; y, s) = \psi_k(x, t)\xi_k(y, s)$$

for some complex function $\xi_k(y, s)$. Since $\psi_k^* = \psi_k\xi_k$ must satisfy the identity (2), one obviously have that

$$(4) \quad \int |\xi_k(y, s)|^2 dy \equiv 1 \quad \text{for every time } s.$$

Nevertheless, not every wave function $\psi^*(x, t; y, s)$ can be decomposed into a product like in (3). Consider for example two wave functions $\psi_k^* = \psi_k\xi_k$ that satisfy (3) for $k = 1, 2$. Given a pair of non-zero constants β_1 and β_2 , it is easy to verify that the superposition $\beta_1\psi_1^* + \beta_2\psi_2^*$ has a product decomposition as in (3) only when there is a constant $\gamma \neq 0$ such that $\psi_2 \equiv \gamma\psi_1$ or $\xi_2 \equiv \gamma\xi_1$.

Similar restrictions on ψ_k and ξ_k can be deduced, when one requests that the superpositions $\beta_1\psi_1^* + \beta_2\psi_2^*$ and $\beta_1\psi_1 + \beta_2\psi_2$ satisfy the identity (2). We must point out that $\beta_1\psi_1^* + \beta_2\psi_2^*$ does not need to have a product decomposition, in order to obtain the new restrictions. Thus, let $\psi_k^* = \psi_k\xi_k$ be two wave functions that satisfy the corresponding Schrödinger equation and the identities (2) and (3) for $k = 1, 2$. The linear combination $\beta_1\psi_1^* + \beta_2\psi_2^*$ obviously satisfy the same Schrödinger equation. Moreover, the following identity easily holds,

$$|\beta_1\psi_1^* + \beta_2\psi_2^*|^2 = |\beta_1\psi_1^*|^2 + 2\Re(\beta_1\psi_1^*\overline{\beta_2\psi_2^*}) + |\beta_2\psi_2^*|^2;$$

and a similar result is obtained by writing ψ_k instead of ψ_k^* . Whence, the linear combinations $\beta_1\psi_1^* + \beta_2\psi_2^*$ and $\beta_1\psi_1 + \beta_2\psi_2$ satisfy the identity (2) if and only if:

$$\Re\left(\beta_1\overline{\beta_2} \int \psi_1^*(x, t; y, s)\overline{\psi_2^*(x, t; y, s)}dy\right) = \Re\left(\beta_1\overline{\beta_2}\psi_1(x, t)\overline{\psi_2(x, t)}\right).$$

One can easily rewrite this equality by recalling equation (3) and introducing the inner product $\langle \xi_1, \xi_2 \rangle_s$ that depends on s ,

$$(5) \quad \Re \left(\beta_1 \overline{\beta_2} \psi_1(x, t) \overline{\psi_2(x, t)} (\langle \xi_1, \xi_2 \rangle_s - 1) \right) = 0,$$

$$(6) \quad \text{where} \quad s \mapsto \langle \xi_1, \xi_2 \rangle_s := \int \xi_1(y, s) \overline{\xi_2(y, s)} dy.$$

We can so conclude that there exists a pair of non-zero constants β_1 and β_2 for which the identities (2) and (5) hold if and only if one of the following conditions is satisfied:

- (7) The inner product $\langle \xi_1, \xi_2 \rangle_s \equiv 1$ for every time s .
- (8) There are two real constants θ_1 and θ_2 such that the functions $e^{-i\theta_1} \psi_1(x, t) \overline{\psi_2(x, t)} \in \mathbb{R}$ and $e^{-i\theta_2} (\langle \xi_1, \xi_2 \rangle_s - 1) \in \mathbb{R}$ take only real values for all positions x and times t and s .

Notice that the latter condition (8) automatically implies that one can find constants β_1 and β_2 such that $e^{i\theta_1+i\theta_2} \beta_1 \overline{\beta_2}$ is purely imaginary, and so the identities (5) and (2) hold.

3. CONCLUSIONS

This result has been proved in the previous chapter: If there exists a pair of wave functions $\psi_k^* = \psi_k \xi_k$ that satisfy the identities (2) and (3) for $k = 1, 2$, then the superpositions $\beta_1 \psi_1^* + \beta_2 \psi_2^*$ and $\beta_1 \psi_1 + \beta_2 \psi_2$ also satisfy the identity (2) if and only if the condition (8) holds or the inner product $\langle \xi_1, \xi_2 \rangle_s \equiv 1$ for every time s . Both conditions obviously impose strong restrictions on the functions ψ_k and ξ_k , so that we can safely conclude that the existence of such wave functions $\psi_k^* = \psi_k \xi_k$ is incompatible with the superposition property and the technique of separation of variables.

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